GINF5020, Spring 2007
Notes for Week of January 8
Mathematical Induction and Recursive Definitions

1 Mathematical Induction

- $P(x)$: a statement (predicate) about x, which can be **true** or **false**, but not both. An example: $P(x)$ is "protein x has a transmembrane domain." Another example: the GC contents of sequence x is more than 60%.

- Today we will mostly use statements about numbers to demonstrate proofs by mathematical induction. Next week we will use induction to prove that the Needleman-Wunsch algorithm finds an optimal global alignment of two sequences.

- To prove $\forall n \geq 1, P(n)$:
  - Basis step (Basis): show that $P(1)$ is true.
  - Induction hypothesis (Hypothesis): assume that $P(n)$ is true.
  - Inductive step (Induction): show that $\forall n \geq 1, (P(n) \rightarrow P(n+1))$ is true.

- Prove: The sum of the first n positive odd numbers is n^2.

  Observations: $n = 1$, $1 = 1^2$; $n = 2$, $1 + 3 = 4 = 2^2$; $n = 3$, $1 + 3 + 5 = 9 = 3^2$.

  The sum of the first n positive odd numbers can be written as

  $$1 + 3 + \ldots + (2n-1) = \sum_{i=1}^{n}(2i-1).$$

  The statement that we want to prove is: $\forall n \geq 1, \sum_{i=1}^{n}(2i-1) = n^2$. The formal proof is as follows:

  Basis: $n = 1$,

  $$\begin{aligned} \sum_{i=1}^{1}(2i-1) &= (2 \cdot 1 - 1) \\ &= 1 \\ &= 1^2. \end{aligned}$$

  Hypothesis: Assume that $\sum_{i=1}^{n}(2i-1) = n^2$ is true.

  Induction:

  $$\sum_{i=1}^{n+1}(2i-1) = \left(\sum_{i=1}^{n}(2i-1)\right) + (2(n+1)-1)$$

$$\begin{aligned} &= n^2 + (2n + 2 - 1) \\ &= n^2 + 2n + 1 \\ &= (n + 1)^2. \end{aligned}$$

- Prove: $\forall n \geq 1$, $n^3 - n$ is divisible by 3.

  Observations: $n^3 - n = n(n^2 - 1) = n(n - 1)(n + 1)$; three consecutive numbers; one of them must be a multiple of 3; so their product must be a multiple of 3. The proof by induction is as follows:

  Basis: $1^3 - 1 = 1 - 1 = 0$. Zero is divisible by 3.

  Hypothesis: Assume $n^3 - n$ is divisible by 3; that is, $n^3 - n = 3m$ for some integer m.

  Induction:

$$\begin{aligned} (n + 1)^3 - (n + 1) &= (n^3 + 3n^2 + 3n + 1) - (n + 1) \\ &= n^3 + 3n^2 + 2n \\ &= (n^3 - n) + 3n^2 + 3n \\ &= 3m + 3n^2 + 3n \\ &= 3(m + n^2 + n), \end{aligned}$$

  which is divisible by 3.

- Prove: $\forall n \geq 1, 1 + 2 + \cdots + n = n(n + 1)/2$.

  What is the sum of $1, 2, \ldots, 100$?

$$\begin{array}{rrrcr} & 1 & 2 & \cdots & 100 \\ + & 100 & 99 & \cdots & 1 \\ \hline & 101 & 101 & \cdots & 101 \end{array}$$

  $100(100 + 1)/2 = 5050$.

  Basis: $1 = 1(1 + 1)/2$.

  Hypothesis: Assume $\sum_{i=1}^{n} i = n(n + 1)/2$.

  Induction:

$$\begin{aligned} \sum_{i=1}^{n+1} i &= \left(\sum_{i=1}^{n} i \right) + (n + 1) \\ &= \frac{n(n + 1)}{2} + (n + 1) \\ &= \frac{n(n + 1)}{2} + \frac{2(n + 1)}{2} \end{aligned}$$

$$
\begin{aligned}
&= \frac{n(n+1)+2(n+1)}{2} \\
&= \frac{(n+1)(n+2)}{2} \\
&= \frac{(n+1)((n+1)+1)}{2}.
\end{aligned}
$$

- Prove: $\forall n \geq 0, 1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$.

  Basis: $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$.

  Hypothesis: Assume $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$.

  Induction:

$$
\begin{aligned}
\sum_{i=0}^{n+1} 2^i &= \left(\sum_{i=0}^{n} 2^i\right) + 2^{n+1} \\
&= (2^{n+1} - 1) + 2^{n+1} \\
&= 2 \cdot 2^{n+1} - 1 \\
&= 2^{n+2} - 1 \\
&= 2^{(n+1)+1} - 1.
\end{aligned}
$$

- $2^n \times 2^n$ chess board with one square removed, can be covered by L-shaped pieces.

- Quiz 1.

2 Recursive Definitions and Recurrence Relations

- One application of mathematical induction is to prove properties of recursively defined things.

- To recursively define a function with the set of nonnegative integers as its domain:

  1. Specify the value of the function at zero.
  2. Give a rule for finding its value at an integer in terms of its values at smaller integers.

- Fibonacci numbers.
  $f(0) = 0,\ f(1) = 1$.
  $f(n) = f(n-1) + f(n-2),\ n \geq 2$.

- $\alpha = (1 + \sqrt{5})/2$.
  Prove: $\forall n \geq 3,\ f(n) > \alpha^{n-2}$.
  α is known as the golden ratio. Read *The Da Vinci Code* by Dan Brown to learn more.

- **Recursive definitions** for sets:
  1. An initial set of elements is given.
  2. Rules are provided to generate new elements from existing elements.
  3. Assert that there are no other elements.

- Let λ be the empty string. To define the set $\Sigma^\star$ of strings over the alphabet Σ:
  $\lambda \in \Sigma^\star$.
  If $w \in \Sigma^\star$ and $a \in \Sigma$, then $wa \in \Sigma^\star$.

- The length of a string w, $l(w)$.
  $l(\lambda) = 0$.
  $l(wa) = l(w) + 1$, if $w \in \Sigma^\star$ and $a \in \Sigma$.

- **Recurrence Relations**

  Given a function defined by a recurrence relation, we are interested in its **closed form**. Usually we first guess the closed form, and then use mathematical induction to prove that our guess is correct.

- The Tower of Hanoi.
  $H(1) = 1$ and $H(n) = 2H(n-1) + 1$.
  Closed form: $H(n) = 2^n - 1$.

- The number of bit strings of length n that do not have two consecutive 0s.
  $f(1) = 2$, $f(2) = 3$, $f(n) = f(n-1) + f(n-2)$.
  Does this recurrence relation look familiar?

- A string of decimal digits is *valid* if it contains an even number of zeros. Let $V(n)$ be the number of valid strings of length n.
  $V(1) = 9$ and $V(n) = 9V(n-1) + (10^{n-1} - V(n-1))$.

- There are n straight lines in general position on a plane. Let $P(n)$ be the number of regions on the plane.
  Recurrence relation: $P(0) = 1$ and $P(n) = P(n-1) + n$.
  Closed form: $P(n) = n(n+1)/2 + 1$.

- There are n planes in general position in the three-dimensional space. Let S be the number of cells in the space.
  $S(0) = 1$ and $S(n) = S(n-1) + P(n-1)$.

- There is a circle and there are n straight lines in general position on a plane; $n \geq 1$. All the lines are tangent to the circle. Let $f(n)$ be the number of intersections. $f(1) = 1$, $f(2) = 3$, $f(3) = 6$, $f(4) = 10$.
  Recurrence relation: $f(n) = f(n-1) + n$.
  Closed form: $f(n) = n(n+1)/2$.

- There is a circle and there are n straight lines in general position on a plane; $n \geq 1$. All the lines are tangent to the circle. Let $f(n)$ be the number of (finite or infinite) regions. $f(1) = 3$, $f(2) = 6$, $f(3) = 10$, $f(4) = 15$.
  Recurrence relation: $f(n) = f(n-1) + n + 1$.
  Closed form: $f(n) = n(n+1)/2 + n + 1$.

- Quiz 2.