FUNDAMENTAL CONCEPTS AND METHODS FOR SYSTEMS MODELING:
A Mathematical Foundation for the Description of Physical, Chemical, and Biological Processes

Outline of Lecture Unit 1

LINEAR VECTOR SPACES:

BASIC CONCEPTS

prepared by

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LINEAR VECTOR SPACES: BASIC CONCEPTS

1 INTRODUCTION

1.1 Motivation for Study

Linear vector space concepts provide the necessary general *mathematical structure:*

- for *unified* treatment of all types of linear problems and their associated equations (algebraic, ordinary differential, partial differential, integral, integrodifferential, functional, etc.),
- for studying general properties of nonlinear problems and their associated equations, and
- for formally deriving *approximations* to problems/equations.

1.2 Spaces: A Preview

1. *Linear vector space:* a set of generalized "vectors" (such as functions and other transformations) that obey certain (addition and multiplication) rules; it is defined over a *field* of numbers (scalars) such as the real or complex numbers.
2. *Metric space:* a set of "points" with a distance function defined.
3. *Normed space:* a linear vector space with a distance (norm).
4. *Euclidean space:* a linear vector space with an inner product. (Complex Euclidean spaces - i.e. defined over the complex numbers field - are also called *unitary* spaces).
5. *Banach space:* a complete (i.e. with "nice" convergence properties) normed space.
6. *Hilbert space:* a Banach space with an inner product.

2 SOME PRELIMINARY CONCEPTS FROM REAL AND COMPLEX ANALYSIS

2.1 Fields

A *field* $\mathcal{F}$ is a set of scalars (numbers) such that for $\alpha, \beta, \gamma \in \mathcal{F}$
(I). $\alpha + \beta \in \mathcal{F}$
(II). $\alpha\beta \in \mathcal{F}$
and the following properties of addition and multiplication hold:
1. $\alpha + \beta = \beta + \alpha, \alpha\beta = \beta\alpha$ (commutativity)
2. $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$, $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ (associativity)
3. $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$ (distributivity)

4. $\exists 0$ and $\exists 1$: $\alpha + 0 = \alpha,\ \alpha 1 = \alpha$
5. $\exists -\alpha$: $-\alpha + \alpha = 0$
6. for $\alpha \neq 0$ $\exists \frac{1}{\alpha}$: $(\frac{1}{\alpha})\,\alpha = 1$

Examples

1. the real numbers
2. the complex numbers
3. $[0, 1]$
4. the rational numbers

Non-Examples

1. the integers
2. the irrational numbers

2.2 Metric Spaces

2.2.1 Definition

A *metric space* is a set with a *distance function,* i.e.
$M = \{u, v, \ldots\}$ is a metric space if $\exists$ real distance function $d(u, v) : \forall u, v, w \in M$
(i) $d(u, v) \geq 0[= 0 \Leftrightarrow u = v]$
(ii) $d(u, v) = d(v, u)$
(iii) $d(u, v) \leq d(u, w) + d(w, v)$

2.2.2 Convergence

1. $u_n \to u$ as $n \to \infty$ if $d(u_n, u) \to 0$ as $n \to \infty (u, u_n \in M)$
2. A sequence $\{u_n\}_1^\infty$ in M is a Cauchy sequence if $\lim_{m,n\to\infty} d(u_m, u_n) = 0$, i.e. $\forall \epsilon > 0\ \exists m_\epsilon$: $d(u_m, u_n) < \epsilon\ \forall m, n > m_\epsilon$
3. M is *complete* if every Cauchy sequence in M has a limit in M

2.3 Topology

Let S be a subset of M.
1. $u_0 \in S$ is an *interior point* if $\exists \delta > 0$: $u \in S \forall \{u \in M : d(u, u_0) < \delta\}$
2. $u_0 \in M$ is an *accumulation point* of S if there is a sequence $\{u_n\}, u_n \in S, u_n \neq u_0$: $\lim_\infty U_n = u_0$
3. S is *open* if if every point in S is an interior point
4. The *closure* $\overline{S}$ of S is the union of S and all its accumulation points
5. S is *closed* if $S = \overline{S}$ or, equivalently, if $M \setminus S$ is open
6. S is *compact* if every sequence in S contains a subsequence that converges in S
7. S is *dense* in M if every point of M is the limit of some sequence in S
8. M is *separable* if it contains a countable dense set

3 LINEAR VECTOR SPACES AND THEIR PROPERTIES

3.1 Vector Spaces

A *linear vector space* $\mathcal{V}$ over or with respect to the field $\mathcal{F}$ is a set of elements (vectors) such that for all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$ and $\alpha, \beta \in \mathcal{F}$
(I). $\exists \mathbf{z} \in \mathcal{V} : \ \mathbf{x} + \mathbf{y} = \mathbf{z}$
(II). $\exists \mathbf{z} \in \mathcal{V} : \ \alpha\mathbf{x} = \mathbf{z}$
and the following properties of vector addition and scalar multiplication hold for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{V}$ and $\alpha, \beta \in \mathcal{F}$:
1. $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ (commutativity)
2. $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z}), \quad \alpha(\beta\mathbf{x})\beta = (\alpha\beta)\mathbf{x}$ (associativity)
3. $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}, \quad (\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$ (distributivity)
4. $\exists$ a unique $\mathbf{0} \in \mathcal{V} : \ \mathbf{x} + \mathbf{0} = \mathbf{x}$
5. $\exists$ a unique $-\mathbf{x} : \ -\mathbf{x} + \mathbf{x} = \mathbf{0}$
6. $1\,\mathbf{x} = \mathbf{x}, \quad 0\,\mathbf{x} = \mathbf{0}, \quad \alpha\,\mathbf{0} = \mathbf{0}$

Examples

1. $\mathcal{R}$, $\mathcal{C}$
2. $\mathcal{R}^n$ (real vectors)
3. the set of square 2×2 matrices
4. the set of real-valued functions $\mathbf{f}(t)$ defined and continuous in $[0, 1]$
5. the set of real-valued functions defined and twice continuously differentiable in $[0, 1]$ that satisfy the o.d.e. $\mathbf{f}''(t) + \mathbf{f}(t) = 0$

Non-Examples

1. the set of real-valued functions defined and twice continuously differentiable in $[0, 1]$ that satisfy the o.d.e. $\mathbf{f}''(t) + \mathbf{f}(t) = 1$
2. the set of all polynomials of second degree with real coefficients

3.2 Subspaces

A nonempty subset $\mathcal{S}$ of $\mathcal{V}$ is a linear space itself if
1. $\mathbf{x}, \mathbf{y} \in ß \Rightarrow \mathbf{x} = \mathbf{y} \in \mathcal{S}$
2. $\mathbf{x} \in \mathcal{S}, \alpha \in \mathcal{F} \rightarrow \alpha\mathbf{x} \in \mathcal{S}$

Examples

1. $\mathcal{V}$ the set of real-valued functions $\mathbf{f}(t)$ defined and continuous in $[0, 1]$; $\mathcal{S}$ the set of all polynomials of degree $\leq$ n
2. $\mathcal{V}$ the set of real-valued functions defined and twice continuously differentiable in $[0, 1]$ that satisfy the o.d.e. $\mathbf{f}''(t) + \mathbf{f}(t) = 0$; $\mathcal{S}$ the set of all functions of the form $\mathbf{f}(t) = \alpha \sin(t)$

3.3 Linear Combinations and Independence

1. $\mathbf{y}$ is a linear combination of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ if $\mathbf{y} = \alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \ldots + \alpha_n\mathbf{x}_n$
2. $\{\mathbf{y} : \mathbf{y} = \alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \ldots + \alpha_n\mathbf{x}_n\}$ is called the linear hull of $(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$
3. $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are

- (i) linearly independent if $\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \ldots + \alpha_n\mathbf{x}_n = \mathbf{0} \Rightarrow \alpha_i = 0, \forall i$
- (ii) linearly dependent if there exist $\alpha_1, \alpha_2, \ldots, \alpha_n$, not all zero, such that $\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \ldots + \alpha_n\mathbf{x}_n = \mathbf{0}$ ($\Leftrightarrow$ some $\mathbf{x}_i$ is a linear combination of the others.)

3.4 Basis and Dimension

$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is a basis of $\mathcal{V}$ and $\mathcal{V}$ is *n-dimensional* if
1. $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent
2. every $\mathbf{x} \in \mathcal{V}$ can be written uniquely as $\mathbf{x} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_n\mathbf{v}_n$

Examples

1. $\mathcal{R}$ is an 1-dimensional l.v.s.
2. $\mathcal{R}^n$ (real vectors) is an n-dimensional l.v.s.
3. the set of square 2×2 matrices is a 4-dimensional l.v.s.
4. the set of all polynomials with real coefficients of degree $\leq$ n is an (n+1)-dimensional l.v.s.
5. the set of real-valued functions $\mathbf{f}(t)$ defined and continuous in $[0, 1]$ is an infinite-dimensional l.v.s.
6. the set of real-valued functions defined and twice continuously differentiable in $[0, 1]$ that satisfy the o.d.e. $\mathbf{f}''(t) + \mathbf{f}(t) = 0$ is a 2-dimensional l.v.s.

3.5 Inner Products and Norms

3.5.1 Inner Product

An *inner or scalar product* $(\mathbf{x} \cdot \mathbf{y})$ [other common notations are $(\mathbf{x}, \mathbf{y}), < \mathbf{x}, \mathbf{y} >, (\mathbf{x}|\mathbf{y})$, etc.] in $\mathcal{V}$ is a function $\mathcal{V} \times \mathcal{V} \rightarrow \mathcal{R}$ such that $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{V},\ \alpha, \beta \in \mathcal{R}$:

- (i) $(\mathbf{x} \cdot \mathbf{y}) = (\mathbf{y} \cdot \mathbf{x})$
- (ii) $(\mathbf{x} \cdot (\alpha\mathbf{y} + \beta\mathbf{z})) = \alpha\,(\mathbf{x} \cdot \mathbf{y}) + \beta\,(\mathbf{x} \cdot \mathbf{z})$
- (iii) $(\mathbf{x} \cdot \mathbf{x}) \geq 0$ $(= 0 \Leftrightarrow \mathbf{x} = \mathbf{0})$

3.5.2 Length (or Norm)

The inner product can be used to define lengths of vectors (norms); e.g. for $\mathcal{F} = \mathcal{R}$
$\parallel \mathbf{x} \parallel = \sqrt{(\mathbf{x} \cdot \mathbf{x})}$, then $\parallel \alpha\mathbf{x} \parallel = |\,\alpha\,| \parallel \mathbf{x} \parallel$

3.5.3 Angle of Two Vectors

The angle θ between $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$ is defined through

$$\cos\theta = \frac{(\mathbf{x} \cdot \mathbf{y})}{\| \mathbf{x} \| \| \mathbf{y} \|}$$

3.5.4 The Cauchy-Schwartz and Triangle Inequalities

$$\| (\mathbf{x} \cdot \mathbf{y}) \| \leq \| \mathbf{x} \| \, \| \mathbf{y} \|$$

$$\| \mathbf{x} + \mathbf{y} \| \leq \| \mathbf{x} \| + \| \mathbf{y} \|$$

If $\cos\theta = 0$ then

$$\| \mathbf{x} + \mathbf{y} \|^2 = \| \mathbf{x} \|^2 + \| \mathbf{y} \|^2$$

(The generalized Pythagorean theorem)

3.5.5 Definition of a Euclidean Space

A linear vector space $\mathcal{E}$ (finite or infinite dimensional: denoted by $\mathcal{E}_n$ and $\mathcal{E}_\infty$ respectively) with an inner product.

3.6 Orthonormal Bases and Orthonormalization

3.6.1 Orthogonality

1. $\mathbf{x}, \mathbf{y} \in \mathcal{E}$ are orthogonal when $(\mathbf{x} \cdot \mathbf{y}) = 0$

2. If $\mathcal{M}$ is a subspace of $\mathcal{E}$ then
$\mathcal{M}^\perp = \{\mathbf{y} \in \mathcal{E} : \; (\mathbf{x} \cdot \mathbf{y}) = 0, \forall \mathbf{x} \in \mathcal{M}\}$
is called the *orthogonal complement* of $\mathcal{M}$.

3.6.2 Orthonormal (ON) Basis: Definition and Properties

1. A basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ of $\mathcal{E}_n$ is called *orthonormal* if

$$(\mathbf{e}_i \cdot \mathbf{e}_j) = \delta_{ij}$$

where δ_{ij} is the Kronecker Delta:

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0 & i \neq j \end{cases}$$

2. If the basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ of $\mathcal{E}_n$ is orthonormal and

$$\mathbf{x} = \sum_{k=1}^{n} x_k \varepsilon_k, \quad \mathbf{y} = \sum_{k=1}^{n} y_k \varepsilon_k$$

then

- (i) $x_k = (\mathbf{x} \cdot \varepsilon_k)$
- (ii) $\| \mathbf{x} \|^2 = \sum_{k=1}^n x_k^2$
- (iii) $(\mathbf{x} \cdot \mathbf{y}) = \sum_{k=1}^n x_k y_k$

3.6.3 Orthogonal Projection

1. For $\mathbf{x} \in \mathcal{E}_n$, $\mathcal{M}$ subspace of $\mathcal{E}_n$:
$\mathbf{x}'$ is the *orthogonal projection* of $\mathbf{x}$ on $\mathcal{M}$ if $\mathbf{x} = \mathbf{x}' = \mathbf{x}''$ with $\mathbf{x}' \in \mathcal{M}$
2. If $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m$ is an ON-basis of $\mathcal{M}$ then the projection $\mathbf{x}'$ of $\mathbf{x}$ on $\mathcal{M}$ is given by

$$\mathbf{x}' = \sum_{k=1}^{m} ((\mathbf{x} \cdot \mathbf{e}_k))\, \mathbf{e}_k$$

Note: $\mathbf{x}'$ is the "best approximation" to $\mathbf{x}$ that one has if "confined" in $\mathcal{M}$: indeed $\| \mathbf{x} - \mathbf{y} \|, \mathbf{y} \in \mathcal{M}$ is minimized for $\mathbf{y} = \mathbf{x}'$

3.6.4 Gram-Schmidt Orthonormalization

Given: a basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ of $\mathcal{E}_n$
Sought: ON-basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ of $\mathcal{E}_n$
Construction: The Gram-schmidt process
STEP 1.

$$\mathbf{e}_1 = \frac{1}{\| \mathbf{v}_1 \|} \mathbf{v}_1$$

STEP 2.

$$\mathbf{f}_2 = \mathbf{v}_2 - ((\mathbf{v}_2 \cdot \mathbf{e}_1))\, \mathbf{e}_1, \quad \mathbf{e}_2 = \frac{1}{\| \mathbf{f}_2 \|} \mathbf{f}_2$$

...
STEP k

$$\mathbf{f}_k = \mathbf{v}_k - ((\mathbf{v}_k \cdot \mathbf{e}_1))\, \mathbf{e}_1 - \ldots - ((\mathbf{v}_k \cdot \mathbf{e}_{k-1}))\, \mathbf{e}_{k-1}, \quad \mathbf{e}_2 = \frac{1}{\| \mathbf{f}_k \|} \mathbf{f}_k$$

Examples - Exercises

1. the Legendre polynomials provide an orthonormal basis for the l.v.s. of polynomials of degree $\leq n$ defined on $[0, 1]$ with inner product $(p \cdot q) = \int_0^1 p(t)\, q(t)\, dt$

2. find an orthonormal basis for the set of all functions defined on $[0, 1]$ with inner product $(f \cdot g) = \int_0^1 f(t)\, g(t)\, dt$, that have the general form $f(t) = \sum_{k=0}^{n} \alpha_k \cos(kt) + \sum_{k=1}^{n} \beta_k \sin(kt)$ (for arbitrary choice of the real scalars α, β)

4 PROBLEMS FOR THE MATERIAL OF LECTURE UNIT 1

Work through the examples listed in the outline.